Bayesian Modeling of Time Trends In Component Reliability Data

Dana Kelly
Idaho National Laboratory
Dana.Kelly@inl.gov
Bayesian Modeling of Time Trends In Component Reliability Data

• **Purpose**
  – Illustrate Bayesian modeling of changing reliability for both nonrepairable and repairable components

• **Objectives**
  – Via examples using WinBUGS, will show how Markov chain Monte Carlo (MCMC) simulation can be used for parametric statistical modeling of cases where component reliability is changing over time
  – Will illustrate approaches to Bayesian model validation
# Example Valve Leakage Data

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Failures</th>
<th>Demands</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>52</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>52</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
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<td>5</td>
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<td>52</td>
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<td>6</td>
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<td>8</td>
<td>9</td>
<td>52</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>52</td>
</tr>
</tbody>
</table>

Model for Number of Leakage Events

- Simplest model is $X \sim \text{binomial}(p, n)$
  - Usual binomial assumption is $p$ constant
- Simplest approach to estimating $p$ is to pool data
- Is this valid?
Is There a Time Trend in $p$?

- Graph appears to indicate increasing trend in time, but there's lots of uncertainty
Possible Models for $p$

- **Constant $p$ (no time trend)**
  - Null hypothesis

- **Models for time trend**
  - Logistic
    - $\text{Logit}(p) = \ln(p/(1 - p)) = a + bt$
  - Probit
    - $\text{Probit}(p) = \Phi^{-1}(p) = a + bt$
Constant p

- Will use Jeffreys prior, but could use other noninformative priors or informative prior if desired
- Pooling data (36 failures in 468 demands) gives posterior mean of 0.08
- 90% interval for $p$ is (0.06, 0.099)
Bayesian Model Validation – A First Vital Aside

• Analysis should not stop with parameter estimates

• In Bayesian framework, “model” comprises
  – Likelihood function
    • How data were generated
  – Prior distribution
    • Uncertainty about parameters

• Bayesian inference sometimes criticized for sensitivity to prior
  – In practice, likelihood function can also be in question

• Need to check both parts of our “model”
Prior Predictive Distribution

- The prior predictive distribution is the denominator of Bayes’ Theorem

\[ f(x) = \int f(x | \theta) \pi_0(\theta) d\theta \]

- Gives probability of observing \( X = x \), unconditional upon any particular value of the parameter(s), \( \theta \)
  - Also called the marginal distribution of \( X \)

- Before observing data, can check reasonableness of prior by calculating probabilities for data we expect to see
  - Small probabilities \( \Rightarrow \) prior not consistent with expected data
  - Sometimes called “preposterior analysis”

- Not defined for improper priors (e.g., Jeffreys prior for Poisson data)
Posterior Predictive Distribution

- Gives conditional probability of seeing a new set of data, $x_{\text{rep}}$, given the old set, $x$
- In symbols

$$f( x_{\text{rep}} | x ) = \int f( x_{\text{rep}} | \theta ) \pi( \theta | x ) d\theta$$

- Posterior predictive distribution is primary tool for Bayesian model validation
  - Focuses on predictive validity of model (prior + data)
Using Posterior Predictive Distribution to Check Model

- Is \( x_{rep} \) in tail of \( f(x_{rep}|x) \)?
  - No \( \Rightarrow \) model OK
  - Yes \( \Rightarrow \) problem with prior and/or likelihood
    - Check prior sensitivity
      - Prior-dominated: sharp prior, sparse data, likelihood function centered away from mode of prior
    - Check appropriateness of likelihood
      - For example, are failures independent?
Summary Statistics From Posterior Predictive Distribution

• In addition to $x_{rep}$, can use summary statistic $T(x)$ (e.g., chi-square)
  
  – If $Pr[T(x_{rep}) \geq T(x_{obs})]$ is small $\Rightarrow$ model has limited validity with respect to replicating observed data

• If model cannot replicate observed data reasonably well, it should definitely not be used for extrapolation or prediction
Bayesian Analog of Chi-Square Statistic

\[ \chi^2 = \left( \frac{x - E(X)}{E(X)} \right)^2 \]

- In frequentist statistics, \( \chi^2 \) is a positive real number calculated from the observed data, which is compared to percentiles of a theoretical chi-square distribution, usually under asymptotic assumptions.

- If “observed” value of \( \chi^2 \) is in upper tail of this distribution, reject model.
  
  - Traditionally compare against 95\(^{th}\) percentile.
Bayesian Analog of Chi-Square Statistic

• In Bayesian framework, both $\chi_{\text{obs}}^2$ and $\chi_{\text{rep}}^2$ have a distribution (not necessarily a chi-square distribution)
  – Distributions are simulated via MCMC
• Compare overlap of distributions
  – If $\Pr(\chi_{\text{rep}}^2 \geq \chi_{\text{obs}}^2)$ is small, model is suspect
    • Will refer to $\Pr(\chi_{\text{rep}}^2 \geq \chi_{\text{obs}}^2)$ as “p-value” in what follows
  – Can use to pick best model among several choices
• Note that Bayesian analog does not require data to be binned, as required in frequentist approach
Using DIC to Compare Relative Fit of Multiple Models

- **Deviance** is defined as \(-2 \times \log(\text{likelihood})\)
  - Measure of goodness of fit
- Take average deviance over posterior:
  \[
  D_{\text{bar}} = -2 \int \ln[f(y | \theta)\pi_1(\theta | y)]d\theta
  \]
  - \(D_{\text{bar}}\) is automatically monitored by WinBUGS node called “deviance”
- Deviance Information Criterion: \(\text{DIC} = D_{\text{bar}} + pD\)
  - \(pD\) is effective number of parameters
  - \(pD = D_{\text{bar}} - D_{\hat{}}\)
  - \(D_{\hat{}}\) is deviance evaluated at posterior mean of parameter(s)
Using DIC to Compare Relative Fit of Multiple Models

- Preferred model is one with smallest DIC
  - Has highest chance of replicating data set
- Models with more parameters (i.e., more complex models) are penalized via $pD$
  - Recall Occam’s Razor/Parsimony Principle
- DIC must be measured on same data
- Must have adequate convergence before estimating DIC!!!
Using DIC to Compare Relative Fit of Multiple Models

• DIC (and even pD) can be negative in some cases
  – DIC negative when density function is > 1
    • Smallest DIC still indicates best fitting model
  – Example: three models with DICs of 10, -3, -9
    • Third model, with DIC = -9, is best fit
  – If pD is negative, cannot use DIC
  – Can use Dbar, but use caution if models have different number of parameters, because there is no penalty for over-parametrizing

• DIC is measure of relative goodness of fit
  – Model with smallest DIC can still be poor fit
Constant p Model for Valve Leakage Probability Revisited

- **Bayesian chi-square goodness-of-fit:**
  - \( p \text{.value} = 0.15 \)

- **Deviance information criterion (DIC) for comparison with logistic and probit models:**

![Deviance information table](image-url)
Logistic Model: \( \text{Logit}(p) = a + bt \)

- *WinBUGS model*

Logistic model for time trend in valve leakage model

```plaintext
for (i in 1:N) {
    x[i] ~ dbin(p[i], n[i]) # Binomial distribution for failures in each year
    logit(p[i]) <- a + b*i # Use of logit() link function for p[i]
}

a~dflat() # Diffuse prior for a
b~dflat() # Diffuse prior for b
```
Multi-Parameter Models: Monitoring Convergence Is Essential (Second Vital Aside)

- **MCMC is both art and science**
- **Must choose initial values for sampling**
  - Best to pick values where posterior density is large
    - **MLEs (or modes of marginal distribution) often a good choice, if available**
  - Care in choosing initial values usually not an issue except for very complicated models
    - Less complicated models will usually converge quickly even with poor initial estimates
Monitoring Convergence – Three Questions

- How to tell when Markov chain has converged
  - How many samples needed for burn in?
    - Markov chain is stationary after burn-in
    - Has sampler obtained good coverage of posterior?
    - How many samples needed for desired precision?
      - Determined by amount of information in prior and data
How Many Samples for Burn In?

- Use History plot in WinBUGS, monitoring from first sample
  - Not enough samples
  - Better
Good Coverage of Posterior?

- Run multiple chains, starting at different points
- Look for good mixing of chains
  - Poor mixing
  - Good mixing
How Many Samples After Convergence?

- *WinBUGS* manual suggests running until Monte Carlo error is 5% or less of sample mean for each parameter.
- None of the three aspects of convergence tends to be a problem for simple models.
  - But do not forget about it!
Convergence Diagnostics - BGR

• Brooks-Gelman-Rubin (BGR) statistic is implemented in WinBUGS
  – Overview
    • Compares estimates of between-chain variance and within-chain variance
      – If chains have converged, all should have same within-chain variance
      – $\text{BGR} \approx 1$ indicates convergence
    • Must run at least 2 chains to calculate BGR statistic
Convergence Diagnostics - BGR

• Details
  – Between-chain estimate is green
  – Within-chain estimate is blue
  – Ratio (between/within) is red
    • Ratio expected to start out > 1 and converge to 1
      – Rule of thumb: $R < 1.2$ for convergence
    • Also want between-chain and within-chain estimates to be stable
    • Double-click on BGR graph, then left click to see precise values of $R$
Convergence Diagnostics - BGR

- Example

- Indicates convergence
Monitoring Convergence - Summary

• For each parameter in the model
  – Run multiple chains and monitor history plots to check for adequate mixing of chains
  – Check BGR plot
  – Check MC error relative to mean value
Logistic Model for Valve Leakage Probability: $\text{Logit}(p) = a + bt$

• **Results**

  - $Pr(b > 0)$ is $> 0.95$
    - Strong evidence of increasing trend in $p$ over time
Logistic Model: \( \text{Logit}(p) = a + bt \)

- **Results for \( p \) in each year:**

  ![Node statistics table](image)

  - **Recall estimate with constant \( p \) (pooled estimate)**
    - **Mean** = 0.08, (0.06, 0.099)
Logistic Model: $\text{Logit}(p) = a + bt$

- *Uncertainty in } p \text{ for each year*
Logistic Model: Logit(p) = a + bt

- Bayesian chi-square goodness-of-fit:
  - p.value = 0.47
- Significantly better than constant-p model (0.15)
- Deviance information criterion (DIC) for comparison with constant-p and probit models:
  - Significantly better than constant-p model (44.26)
Predicting $p$ In Future Years

- Can use MCMC to predict distribution of $p$ in future years
- Example: What is $p$ in the $10^{th}$ year?
- Code: $\text{logit}(p[10]) \leftarrow a + b*10$
Predicting p In Future Years

• Results

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
</tr>
</thead>
<tbody>
<tr>
<td>p[1]</td>
<td>0.04171</td>
<td>0.01501</td>
<td>3.214E-4</td>
<td>0.02066</td>
<td>0.03981</td>
<td>0.06913</td>
</tr>
<tr>
<td>p[2]</td>
<td>0.04749</td>
<td>0.01429</td>
<td>3.008E-4</td>
<td>0.02662</td>
<td>0.04611</td>
<td>0.07508</td>
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<tr>
<td>p[3]</td>
<td>0.05427</td>
<td>0.01344</td>
<td>2.691E-4</td>
<td>0.03405</td>
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<td>0.07767</td>
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<tr>
<td>p[4]</td>
<td>0.06221</td>
<td>0.01262</td>
<td>2.232E-4</td>
<td>0.04285</td>
<td>0.06147</td>
<td>0.09414</td>
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<tr>
<td>p[5]</td>
<td>0.07151</td>
<td>0.01232</td>
<td>1.609E-4</td>
<td>0.05266</td>
<td>0.07086</td>
<td>0.09283</td>
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<td>p[6]</td>
<td>0.0824</td>
<td>0.01336</td>
<td>9.171E-5</td>
<td>0.06146</td>
<td>0.06171</td>
<td>0.1057</td>
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<tr>
<td>p[7]</td>
<td>0.09513</td>
<td>0.01665</td>
<td>1.138E-4</td>
<td>0.06919</td>
<td>0.09442</td>
<td>0.1238</td>
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<tr>
<td>p[8]</td>
<td>0.1099</td>
<td>0.02255</td>
<td>2.582E-4</td>
<td>0.07524</td>
<td>0.1067</td>
<td>0.1491</td>
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<tr>
<td>p[9]</td>
<td>0.127</td>
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<td>0.1247</td>
<td>0.1819</td>
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<tr>
<td>c[T01]</td>
<td>0.1466</td>
<td>0.04214</td>
<td>7.221E-4</td>
<td>0.08602</td>
<td>0.1429</td>
<td>0.2222</td>
</tr>
</tbody>
</table>
Probit Model: $\Phi^{-1}(p) = a + bt$

- *WinBUGS model*

```plaintext
Probit model for time trend in valve leakage model
{
    for (i in 1:7) {
        x[i] ~ dbin(p[i], n[i]) #Binomial distribution for failures in each year
        probit(p[i]) <- a + b*i #Use of probit() link function for p[i]
    }
    a~dflat() #Diffuse prior for a
    b~dflat() #Diffuse prior for b
}
```
Probit Model: $\Phi^{-1}(p) = a + bt$

- **Results**

  ![Node statistics](image1)

  ![Posterior density](image2)

  - Again, $Pr(b > 0)$ is close to unity
    - Strong evidence of increasing trend in $p$
Probit Model: $\Phi^{-1}(p) = a + bt$

- Results for $p$ in each year:
Probit Model: $\Phi^{-1}(p) = a + bt$

- Uncertainty in $p$ for each year
Probit Model: $\Phi^{-1}(p) = a + bt$

- Bayesian chi-square goodness-of-fit:
  - p.value = 0.46
- Essentially same as logit model
- Deviance information criterion (DIC) for comparison with constant-p and logit models:

```
Deviance information

<table>
<thead>
<tr>
<th></th>
<th>Dbar</th>
<th>Dhat</th>
<th>DIC</th>
<th>pD</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>35.97</td>
<td>33.94</td>
<td>38.0</td>
<td>2.032</td>
</tr>
<tr>
<td>total</td>
<td>35.97</td>
<td>33.94</td>
<td>38.0</td>
<td>2.032</td>
</tr>
</tbody>
</table>
```

- Equivalent to logit model
Conclusions

- Strong evidence of increasing trend in $p$
  - $p$-value for logit and probit models significantly better than model with constant $p$
  - Slight analyst preference for logistic model over probit model
- Can specify distribution for $p$ in each year as logistic-normal
  - Logistic-normal distribution can be interpreted as constrained lognormal distribution (Kelly, 1992)
- Ignoring trend gives overly confident estimate of $p$
  - Especially a problem for extrapolating to future years
Modeling Time Trends In $\lambda$

- Will use example data supplied by Andrei Rodionov
- Data are in form of number of failures in each of 20 years
  - Have data on exposure time for each year
- Number of failures ($x_i$) in each year will be Poisson with parameter $\lambda_it_i$
- $\lambda$ may be a function of time
  - Leads to non-homogeneous Poisson process (NHPP)
## Example Data – Component 13

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Failures</th>
<th>Exposure Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>176.046</td>
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<tr>
<td>2</td>
<td>0</td>
<td>244.472</td>
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<tr>
<td>3</td>
<td>0</td>
<td>314.672</td>
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<tr>
<td>4</td>
<td>1</td>
<td>371.008</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>434.02</td>
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<tr>
<td>6</td>
<td>0</td>
<td>457.732</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>499.988</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>581.386</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>571.62</td>
</tr>
<tr>
<td>10</td>
<td>2.5</td>
<td>533.176</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>527.394</td>
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<tr>
<td>12</td>
<td>1</td>
<td>492.86</td>
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<tr>
<td>13</td>
<td>0</td>
<td>440.062</td>
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<td>3</td>
<td>371.584</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>301.392</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>245.052</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>182.064</td>
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<td>18</td>
<td>1</td>
<td>156</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>102.156</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>9.036</td>
</tr>
</tbody>
</table>
Graphical Check for Time Trend in $\lambda$

![Caterpillar plot for $\lambda$](image)
Need Model for $\lambda(t)$

- Many possibilities:
  - Constant: $\lambda(t) = \lambda_o$
  - Linear: $\lambda(t) = \lambda_o + at$
  - Log-linear: $\ln[\lambda(t)] = a + bt$
  - Power law: $\lambda(t) = (\alpha/\beta)(t/\beta)^{\alpha-1}$
  - Extended power law: $\lambda(t) = \alpha\lambda_o t^{\alpha-1} + \lambda_{\text{ext}}$

- No theoretical justification for any of these
- Will illustrate first three in WinBUGS
WinBUGS Script

Models for time trend in lambda model
{
    for (i in 1:N) {
        #lambda[i] <- lambda.zero #Constant model
        #log(lambda[i]) <- a + b*i #Loglinear model
        #lambda[i] <- alpha/beta*pow(i/beta, alpha -1) #Power law model
        #lambda[i] <- lambda.zero + a*i #Linear model
        lambda[i] ~ dgamma(0.5, 0.0001) #Used for constructing waterfall plot
        mu[i] <- lambda[i]*s[i]
        x[i] ~ dpois(mu[i])
        x.rep[i] ~ dpois(mu[i])
        diff.obs[i] <- pow(x[i] - mu[i], 2)/mu[i]
        diff.rep[i] <- pow(x.rep[i] - mu[i], 2)/mu[i]
    }
    chisq.obs <- sum(diff.obs[])
    chisq.rep <- sum(diff.rep[])
    p.value <- step(chisq.rep - chisq.obs)
    #lambda.zero ~ dgamma(0.0001, 0.0001) #Diffuse prior on lambda.zero
    #a ~ dflat() #Diffuse prior on a
    #b ~ dflat() #Diffuse prior on b
    #alpha ~ dexp(1) #Diffuse prior on alpha
    #beta ~ dgamma(0.0001, 0.0001) #Diffuse prior on beta
}
Constant Model $\lambda(t) = \lambda_0$

- **Results**

<table>
<thead>
<tr>
<th>Node statistics</th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>lambda.zero</td>
<td>0.002987</td>
<td>6.521E-4</td>
<td>4.46E-6</td>
<td>0.002</td>
<td>0.002846</td>
<td>0.004129</td>
<td>1001</td>
<td>20000</td>
</tr>
</tbody>
</table>

![Node statistics](image1)

![Posterior density](image2)
Constant Model $\lambda(t) = \lambda_0$

- $p.val = 0.165$
- $DIC$ (for comparison with other models)
Linear Model: $\lambda(t) = \lambda_0 + at$

- Results
Linear Model: $\lambda(t) = \lambda_0 + at$

- **Results**

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1.528E-4</td>
<td>1.092E-4</td>
<td>5.094E-6</td>
<td>-2.948E-5</td>
<td>1.655E-4</td>
<td>3.32E-4</td>
<td>4001</td>
<td>14000</td>
</tr>
</tbody>
</table>

- **Significant** $Pr(a > 0) \Rightarrow$ aging
Linear Model: $\lambda(t) = \lambda_0 + at$

- $p.value = 0.14$
- $DIC$
- Equivalent to simpler model with constant $\lambda$
Linear Model: $\lambda(t) = \lambda_0 + at$

- Results for $\lambda$ in each year:

<table>
<thead>
<tr>
<th>Node</th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>lambda[1]</td>
<td>0.001797</td>
<td>9.967E-4</td>
<td>4.288E-5</td>
<td>4.882E-4</td>
<td>0.001628</td>
<td>0.003697</td>
<td>22001</td>
<td>20000</td>
</tr>
<tr>
<td>lambda[2]</td>
<td>0.001956</td>
<td>9.133E-4</td>
<td>3.54E-5</td>
<td>7.944E-4</td>
<td>0.001794</td>
<td>0.003679</td>
<td>22001</td>
<td>20000</td>
</tr>
<tr>
<td>lambda[3]</td>
<td>0.002102</td>
<td>8.367E-4</td>
<td>3.417E-5</td>
<td>0.001013</td>
<td>0.001861</td>
<td>0.003685</td>
<td>22001</td>
<td>20000</td>
</tr>
<tr>
<td>lambda[4]</td>
<td>0.002255</td>
<td>7.668E-4</td>
<td>3.09E-5</td>
<td>0.001243</td>
<td>0.002129</td>
<td>0.003733</td>
<td>22001</td>
<td>20000</td>
</tr>
<tr>
<td>lambda[5]</td>
<td>0.002408</td>
<td>7.122E-4</td>
<td>2.593E-5</td>
<td>0.001441</td>
<td>0.002297</td>
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Linear Model: $\lambda(t) = \lambda_0 + at$

- Uncertainty in lambda for each year
Loglinear Model: $\ln[\lambda(t)] = a + bt$

- **Results**

- $Pr(b > 0)$ is large $\Rightarrow$ aging
- Posterior distribution of $b$ is approximately normal
Loglinear Model: $\ln[\lambda(t)] = a + bt$

- $p\text{.value} = 0.12$
- $DIC$

- Equivalent to simpler model with constant $\lambda$
Loglinear Model: \( \ln[\lambda(t)] = a + bt \)

- Results for lambda in each year:
Loglinear Model: $\ln[\lambda(t)] = a + bt$

- Uncertainty in lambda for each year
Power Law Model: $\lambda(t) = \left(\frac{\alpha}{\beta}\right)(t/\beta)^{\alpha-1}$

- **Results**

- $Pr(\alpha > 1)$ near 0.5 $\Rightarrow$ no evidence for aging
Power Law Model: \( \lambda(t) = (\alpha/\beta)(t/\beta)^{\alpha-1} \)

- \( p\text{-value} = 0.11 \)
- \( DIC \)

<table>
<thead>
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- Negative value for \( pD \)
  - Cannot use DIC in this case
  - Comparing Dbar indicates power-law model is slightly worse than linear or log-linear model
Power Law Model: \( \lambda(t) = \alpha \lambda_0 t^{\alpha - 1} \)

- Results for lambda in each year:

<table>
<thead>
<tr>
<th>lambda</th>
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<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
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Power Law Model: $\lambda(t) = \alpha \lambda_0 t^{\alpha - 1}$

- Uncertainty in lambda for each year
Conclusions for Component 13

- Linear and log-linear models show slight evidence for aging
- Constant-\(\lambda\) model fits slightly better than either of these
- Power-law model suggests \(\lambda\) constant
  - Cannot use DIC in this case because \(pD\) negative
- Overall conclusion: no strong evidence against model with constant \(\lambda\)
Component 13 – First Ten Years Removed

- Models situation where repair occurs at 10 years
  - Component repaired to “as good as new”

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Graphical Check for Time Trend in $\lambda$
Constant Model $\lambda(t) = \lambda_0$

- **Results**

![Node statistics table](image)

![Posterior density graph](image)
Constant Model $\lambda(t) = \lambda_0$

- $p\text{.value} = 0.17$
- $DIC$ (for comparison with other models)
Linear Model: $\lambda(t) = \lambda_0 + at$

- Results
Linear Model: $\lambda(t) = \lambda_0 + at$

- **Results**

  ![Node statistics](image)

  ![Posterior density](image)

- **Significant** $Pr(a > 0) \Rightarrow$ aging
Linear Model: $\lambda(t) = \lambda_0 + at$

- $p.value = 0.52$
  - Indicates good absolute fit of linear model
- DIC

- Better relative fit than simpler model with constant $\lambda$
Linear Model: $\lambda(t) = \lambda_0 + at$

- Results for lambda in each year:

<table>
<thead>
<tr>
<th>Node</th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
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</table>
Linear Model: $\lambda(t) = \lambda_0 + at$

- Uncertainty in lambda for each year
Predicted $\lambda$ From Linear Model

- Results for $\lambda$ in 11th year
Loglinear Model: \( \ln[\lambda(t)] = a + bt \)

- **Results**

  - \( \Pr(b > 0) \) is large \( \Rightarrow \) aging
  - *Posterior distribution of* \( b \) *is approximately normal*
Loglinear Model: $\ln[\lambda(t)] = a + bt$

- $p\.value = 0.32$
- $DIC$

![Deviance information](image)

- Better fit than simpler model with constant $\lambda$
- Not quite as good as linear model
Loglinear Model: \( \ln[\lambda(t)] = a + bt \)

- Results for lambda in each year:
Loglinear Model: $\ln[\lambda(t)] = a + bt$

- Uncertainty in lambda for each year
Predicted \( \lambda \) From Loglinear Model

- Results for \( \lambda \) in 11\(^{th} \) year

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<th>MC_error</th>
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<td>lambda[9]</td>
<td>0.01302</td>
<td>0.006936</td>
<td>2.508E-4</td>
<td>0.004329</td>
<td>0.01172</td>
<td>0.02625</td>
</tr>
<tr>
<td>lambda[10]</td>
<td>0.01659</td>
<td>0.01222</td>
<td>4.988E-4</td>
<td>0.004794</td>
<td>0.01576</td>
<td>0.04174</td>
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<tr>
<td>lambda[11]</td>
<td>0.02701</td>
<td>0.02185</td>
<td>9.444E-4</td>
<td>0.005246</td>
<td>0.02113</td>
<td>0.06679</td>
</tr>
</tbody>
</table>
Power Law Model: $\lambda(t) = (\alpha/\beta)(t/\beta)^{\alpha-1}$

- **Results**

- $Pr(\alpha > 1)$ significant $\Rightarrow$ positive evidence for aging
Power Law Model: \( \lambda(t) = (\alpha/\beta)(t/\beta)^{\alpha-1} \)

- \( p\.value = 0.42 \)
  - Indicates good absolute fit of power-law model

- DIC
  - Negative value for \( pD \)
    - Cannot use DIC in this case
    - \( Dbar \) comparable to linear model
Power Law Model: $\lambda(t) = \alpha \lambda_0 t^{\alpha-1}$

- Results for lambda in each year:

![Node statistics table]

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>lambda[1]</td>
<td>0.001259</td>
<td>0.001084</td>
<td>2.353E-5</td>
<td>1.603E-4</td>
<td>9.483E-4</td>
<td>0.003416</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[2]</td>
<td>0.001985</td>
<td>0.001058</td>
<td>1.885E-5</td>
<td>5.945E-4</td>
<td>0.001808</td>
<td>0.003965</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[3]</td>
<td>0.00274</td>
<td>0.001067</td>
<td>1.159E-5</td>
<td>0.001209</td>
<td>0.00262</td>
<td>0.004674</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[4]</td>
<td>0.003541</td>
<td>0.001167</td>
<td>5.042E-6</td>
<td>0.001851</td>
<td>0.00322</td>
<td>0.00566</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[5]</td>
<td>0.004392</td>
<td>0.001426</td>
<td>1.55E-5</td>
<td>0.002345</td>
<td>0.004237</td>
<td>0.006977</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[6]</td>
<td>0.005297</td>
<td>0.001866</td>
<td>3.289E-5</td>
<td>0.002662</td>
<td>0.005074</td>
<td>0.0087</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[7]</td>
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<td>0.00248</td>
<td>5.375E-5</td>
<td>0.002865</td>
<td>0.005895</td>
<td>0.01085</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[8]</td>
<td>0.007272</td>
<td>0.00326</td>
<td>7.863E-5</td>
<td>0.003013</td>
<td>0.006715</td>
<td>0.0134</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[9]</td>
<td>0.008346</td>
<td>0.004202</td>
<td>1.075E-4</td>
<td>0.003124</td>
<td>0.007527</td>
<td>0.01628</td>
<td>10001</td>
<td>200000</td>
</tr>
<tr>
<td>lambda[10]</td>
<td>0.009477</td>
<td>0.005308</td>
<td>1.404E-4</td>
<td>0.003211</td>
<td>0.008315</td>
<td>0.01955</td>
<td>10001</td>
<td>200000</td>
</tr>
</tbody>
</table>
Power Law Model: $\lambda(t) = \alpha \lambda_0 t^{\alpha-1}$

- Uncertainty in lambda for each year
Predicted $\lambda$ From Power-Law Model

- Results for $\lambda$ in 11th year
Conclusions for Component 13 With First Ten Years Removed

• Significant evidence of aging
• Linear and power-law models provide best fit
  – Log-linear model also provides adequate fit
Modeling Repairable Components

- In this case, component is repaired when failure is detected
- Three potential situations:
  - Repairs tend to improve component over time (reliability growth, Ascher’s “happy” system)
  - Repairs maintain failure rate at essentially constant value
  - Repairs ineffective at preventing aging (Ascher’s “sad” system)
    - In first and third case, observed failure times are not independent and identically distributed – not from a renewal process
    - Cannot simply fit distribution to data
    - Will handle these situations via non-homogeneous Poisson process (NHPP)
Example Data for Compressor Failures

- Cumulative times at which compressor failed were recorded
  - Time to repair compressor neglected
- 90 times were recorded
  - Contained in file “full compressor data.txt”
  - Times are recorded in days

Example Data for Compressor Failures

- Plot $n(t)$ vs. $t$
  - Expect straight line if failure rate is constant
  
- Plot appears slightly concave
  - Reliability growth?

![Graph showing number of failures as a function of time]
Nonhomogeneous Poisson Process

- Recall that one of the assumptions leading to the Poisson distribution was that \( \lambda \) is constant.
- Can relax this assumption, and allow \( \lambda \) to vary with time.
- Leads to what is called a nonhomogeneous Poisson process.
  - Like a Poisson distribution, but with parameter

\[
\mu(t) = \int_0^t \lambda(s) \, ds
\]
Need Model for $\lambda(t)$: Rate of Occurrence of Failures (ROCOF)

- **Will consider four possibilities:**
  - **Constant:** $\lambda(t) = \lambda$
    - *Null hypothesis*
  - **Power law:** $\lambda(t) = (\alpha/\beta)(t/\beta)^{\alpha-1}$
    - *Note that $\alpha = 1$ corresponds to constant $\lambda$*
  - **Linear law:** $\lambda(t) = \lambda_o + at$
    - *$a = 0$ corresponds to constant $\lambda$*
  - **Loglinear law:** $\lambda(t) = \lambda_o e^{bt}$
    - *$b = 0$ corresponds to constant $\lambda$*
Expected Number of Failures In NHPP

- Let \( n(t) \) be number of failures that have occurred in interval \([0, t]\)

\[
E[n(t)] = \mu(t) = \int_{0}^{t} \lambda(s)ds
\]
Relationship To Poisson Process

- \( n(t) \sim \text{Poisson}[\mu(t)] \)
  - Homogeneous Poisson process (HPP) for \( \mu(t) = \lambda t \)
  - Nonhomogeneous otherwise
    - \( \mu(t) = (t/\beta)^\alpha \) for power law
    - \( \mu(t) = \lambda_0 t + (at^2)/2 \) for linear law
    - \( \mu(t) = (\lambda_0/b)(e^{bt} - 1) \) for loglinear law
Relationship To Poisson Process

• Will compare HPP to power-law, linear-law, and loglinear law processes
  – Will use Bayesian analog of chi-square statistic for absolute measure of fit
    • Calculated from posterior predictive distribution
  – Deviance information criterion (DIC) will be measure of relative fit of each model
WinBUGS Implementation

- **WinBUGS model (will show loglinear model later)**

Models for compressor failure with repair model

```c
{  
    #mu[i] <- pow(t[i]/beta, alpha) #Power-law model
    #mu[i] <- lambda.constant*t[i] #Constant model
    mu[i] <- lambda.zero*t[i] + a*pow(t[i], 2)/2 #Linear model
    n[i] ~ dpois(mu[i])  
}  
#alpha ~ dexp(1) #Diffuse priors
#beta ~ dgamma(0.0001, 0.0001)
lambda.zero ~ dgamma(0.0001, 0.0001)
lambda.constant ~ dgamma(0.0001, 0.0001)
a ~ dflat()
}
```
WinBUGS Implementation

- *Script for Bayesian chi-square test*

```r
for(i in 1:M) {
  n.rep[i] ~ dpois(mu[i])  # Value from posterior predictive distribution
  diff.obs[i] <- pow(n[i] - mu[i], 2)/mu[i]
  diff.rep[i] <- pow(n.rep[i] - mu[i], 2)/mu[i]
}
chisq.obs <- sum(diff.obs[])
chisq.rep <- sum(diff.rep[])
p.value <- step(chisq.rep - chisq.obs)
}
```
Results for Exponential Model (HPP)

- Failure rate ($\lambda$):

![Node statistics table]

![Posterior density graph]
Results for Exponential Model (HPP)

- \( p\text{-value} = 1.5 \times 10^{-4} \)
  - Very small value indicates poor fit of HPP
- DIC (for comparison with NHPP)

![Deviance Information Table]

<table>
<thead>
<tr>
<th></th>
<th>Dbar</th>
<th>Dhat</th>
<th>DIC</th>
<th>pD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>575.5</td>
<td>574.5</td>
<td>578.5</td>
<td>1.0</td>
</tr>
<tr>
<td>total</td>
<td>575.5</td>
<td>574.5</td>
<td>578.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Results for Power-Law Process

- **Shape parameter ($\alpha$):**
  - $Pr(\alpha < 1)$ is large
  - $\alpha < 1 \Rightarrow$ rate of increase in $n(t)$ decreases with increasing $t$
  - Compressor is getting better with time!
Results for Power-Law Process

- Scale parameter ($\beta$):

```
<table>
<thead>
<tr>
<th></th>
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<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>beta</td>
<td>22.6</td>
<td>3.808</td>
<td>0.2594</td>
<td>16.43</td>
<td>22.47</td>
<td>28.97</td>
<td>1001</td>
<td>20000</td>
</tr>
</tbody>
</table>
```
Results for Power-Law Process

- $p$-value = 0.82
- $DIC$

- Significantly less than $DIC$ for HPP $\Rightarrow$ power-law process is better at replicating observed data
Results for Linear Law

- $\lambda_0$:
Results for Linear Law

- $a$:

<table>
<thead>
<tr>
<th>Node statistics</th>
<th>mean</th>
<th>sd</th>
<th>MC_error</th>
<th>val5.0pc</th>
<th>median</th>
<th>val95.0pc</th>
<th>start</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-2.139E-6</td>
<td>3.118E-7</td>
<td>1.7E-8</td>
<td>-2.647E-6</td>
<td>-2.143E-6</td>
<td>-1.812E-6</td>
<td>6001</td>
<td>20000</td>
</tr>
</tbody>
</table>

- $Pr(a < 0)$ is large $\Rightarrow$ reliability growth
Results for Linear Law

- $p$-value $= 2.5 \times 10^{-4}$
  - Linear model is poor fit

- DIC
  - Better than constant model, but worse than power-law model
Modeling Loglinear Law

- Recall model: $\lambda(t) = \lambda_o e^{bt}$
- Leads to $\mu(t) = (\lambda_o/b)(e^{bt} - 1)$
  - Parameter $b$ can take on any real value in theory
  - Values very near zero cause numerical difficulties
  - Pragmatic solution
    - Run two cases
      - Case 1: $\text{uniform}(-1, 0)$ prior on $b$ (reliability growth)
      - Case 2: $\text{uniform}(0, 1)$ prior on $b$ (aging)
Loglinear Law – Case 1 (Reliability Growth)

- Results
Loglinear Law – Case 1 (Reliability Growth)

- $p$-value $= 2.8 \times 10^{-4}$
  - Poor fit
- DIC
  - About the same as linear law, worse than power law
Loglinear Law – Case 2 (Aging)

- **Could not run this case**
  - *Undefined real result traps occurred in WinBUGS with uniform(0, 1) prior on b*
  - *Data suggest negative value for b*
  - *Sampling pushes b toward zero in attempt to reach negative values*
  - *Small values of b in denominator cause machine overflow*
Conclusions About Compressor Failure Data

- Data indicate reliability growth is occurring (decreasing failure rate)
- Linear model shows evidence for reliability growth but is poor at replicating observed data
- Power-law model shows evidence for reliability growth and is good at replicating observed data
- Loglinear model suggests reliability growth but is poor fit to data
  - Difficulty implementing this model in WinBUGS
Overall Conclusions

• Bayesian analysis of time-dependent reliability (with and without repair) is feasible
• MCMC simulation is easy with WinBUGS/OpenBUGS
  – WinBUGS is freely available, open-source software
• Can handle variety of models
• Allows for model validation, a crucial component of the analysis
  – Compare absolute fit with Bayesian analog of chi-square statistic and relative fit with DIC
• Can factor in prior information regarding model parameters if desired
Suggested Future Work

• Investigate better ways of implementing loglinear model in WinBUGS

• Investigate nonparametric hazard models within the Bayesian framework
  – Can handle non-monotonic failure intensity

• Investigate Bayesian time-series models
  – Cyclic data

• Benchmark methods against more datasets